A Coalition Game on Finite Groups

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Abstract of Report Talk: This is an initial investigation into possible connections between the mathematical theories of groups and coalition games. An example of a group is the set of symmetries of a square: (0-, 90-, 180-, and 270-degree rotations and flips across the four lines of symmetries) with composition of motions as the binary operation. A coalition game is a set of players and a numerical worth for each coalition (a nonempty subset of players), and an allocation divides the worth of the all-player coalition among the players. Given an allocation, the excess of a coalition is the sum of their payoffs minus the worth of the coalition, which is one way to quantify how happy the coalition is with the allocation. The prenucleolus is an allocation that maximizes the minimum coalition excess. One coalition game on a group uses the group elements as the players, and the worth of a coalition is the number of elements in the subgroup generated by the coalition. For any such coalition game, the prenucleous payoff is shown to be nonnegative for each player and zero for the identity element player. The prenucleolus for two infinite classes of groups were determined: the symmetries of a regular n-gon and the integers 0, 1, ..., n - 1 using addition modulo n as the binary operation.

Received: August 12, 2020

Proportional Network Connectivity and Applications

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Abstract of Report Talk: Network connectivity is often looked at through the parameters of vertex connectivity and edge connectivity, which are the minimum number of vertices (respectively edges) that must be deleted from a graph to disconnect it. We examine vertex and edge connectivity such that to be in a failure state (disconnected), all components of the graph must have order at most $r \cdot n$, where r is a proportion and n is the order of the original graph. We determine the Component Order Vertex (resp. Edge) Proportion Connectivity for basic graph classes. Additionally, we extend the results to the random graph class $G(n, m)$.

Received: August 12, 2020

Quadratization of Scalar Polynomial ODEs

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Abstract of Report Talk: Quadratization is an ad-hoc mathematical process that reduces well-defined scalar polynomial ODEs into quadratic form via the introduction of well-defined auxiliary variables. Formally, we say that a scalar polynomial ODE $\dot{x} = p(x)$ is quadratized by auxiliary variables $z_1 := z_1(x), z_2 := z_2(x), \ldots, z_n := z_n(x)$ if $\dot{x}$ and the Lie derivatives of our auxiliary variables can be written as quadratic polynomials in $z_1, z_2, \ldots, z_n, x$. Quadratization helps us to solve optimization problems involving a large degree polynomial more conveniently as there are already well-established methods of solving quadratic polynomials. Thus, as a result, quadratization has widespread applicability in fields, such as quantum mechanics, number theory, integer factorization, graph theory, and computer vision problems. Quadratization has also been used in the study of nonlinear model order reduction. In our research, we aim to study the quadratization of holomorphic, complex-valued functions of continuous variables. In particular, we study the question of âwhat is the form of a scalar polynomial ODE $\dot{x}$ such that it can be quadratized with precisely one auxiliary variable.â In order to approach this question, our experimentation involved computer-programmed Grobner Bases, which aided us in identifying patterns in the restrictions on smaller degree ODEs. Our main result is that a scalar polynomial ODE $\dot{x}$ of at least degree 5 can be quadratized using exactly one auxiliary variable if and only if some linear transformation of $\dot{x}$ is of the form $x^n + ax^2 + bx$ and the form of the auxiliary variable must be $z := (x - c)^{n-1}$. We also show the relation between monomial quadratization and the [2]-sumset cover problem.

Received: August 12, 2020

Enumerating "Good" Permutations

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Abstract of : Two permutations π and σ are said to be order isomorphic if they are equivalent after pattern reduction. We call a permutation "good" if the first ℓ entries are order isomorphic to the last ℓ entries. Given a k, we wish to enumerate all good permutations on $[k]$ which overlap consecutively. We do this for whenever $\ell \leq k/2$, and via experimentation we conjecture that whenever $\ell > k/2$ the number of good permutations is polynomial in k. We also make a connection of enumerating good permutations to the problem of explicitly determining the expected number of distinct permutation patterns contained in a random permutation.

Received: August 12, 2020

Topological Properties of almost Abelian Lie groups and Homogeneous Spaces

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Abstract of : An almost Abelian Lie group G is a Lie group with a codimension one Abelian subgroup. We begin by studying the topological properties of an almost Abelian Lie group G and its connected subgroups. We establish that an almost Abelian Lie group G is never compact. We then describe explicit conditions for a Lie subgroup of G to be compact and classify all such subgroups. Once classified, we identify the maximal subgroups. It follows from a well-known theorem, the maximal subgroups give the homotopy type of G.
Next we consider homogeneous spaces G/H where H is a closed Lie subgroup. We first show that the fundamental group $\pi_1(G/H)$ is strongly dependent upon the component group $\pi_0(H) = H/H_0$, which is a discrete subgroup of G. This leads us to the classification of all discrete subgroups. Subsequently, we are able to reduce the compactness of G/H to a condition involving H_0, H/H_0 and $\pi_1(G)$. Finally, the homotopy type of G/H is given by the quotient of a maximal subgroup of G by H.
Solvmanifolds for non-nilpotent groups are known to be difficult to study explicitly. The purpose of our study is to provide infinitely many such examples with diverse properties as well as to study their topology. The topology is interesting on its own, and also provides beneficial insights for harmonic analysis and spectral theory.

Received: August 12, 2020

Riemannian Geometry of Almost Abelian Groups

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Abstract of Report Talk: An almost Abelian group is a non-Abelian Lie group with a codimension one Abelian subgroup. We investigate the Riemannian geometry of finite dimensional almost Abelian groups equipped with a left-invariant Riemannian metric by faithful matrix representations. This geometric study covers the Riemannian curvature tensor, Ricci tensor, Riemannian curvature scalar, and the geodesic equation, and we are currently investigating the distance formula between two points. We find ways to derive all details in explicit form by carefully selecting optimal group coordinates and using insights from the theory of linear dynamical systems. These objects provide an infinite supply of explicitly tractable Riemannian homogeneous spaces and serve as a starting point for further studies in geometrical PDEs and space-frequence analysis.

Received: August 12, 2020

The Sunflower Problem

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Abstract of Report Talk: A *sunflower with p petals* consists of p sets whose pairwise intersections are all the same set. The goal of the sunflower problem is to find the smallest $r = r(p, k)$ such that every family of at least r^k k-element sets must contain a sunflower with p petals. Major breakthroughs within the last year show that $r = O(p \max\{\log p, \log k\})$ suffices. In this presentation, we go over the proof strategy used in recent papers and the applications of the sunflower problem to computer science. We reach a bound of $r = O(p \max\{(\log p)^2, \log k\})$ without using the information theoretic results used in other proofs. Additionally, we show that all current techniques are not sufficient to improve the best known bounds, as we exhibit a family with $r = O(p \max\{\log p, \log k\})$ that does not satisfy a key lemma of all recent sunflower papers.

Received: August 12, 2020

Arithmetic Complexity of Domains and the Bergman Projection

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Abstract of Report Talk: For a domain $\Omega \subseteq \mathbb{C}^n$ the *Bergman projection* is the orthogonal projection from $L^2(\Omega)$ to the *Bergman space*, $A^2(\Omega)$, of holomorphic square-integrable functions on Ω. We consider a large class of *singular Reinhardt domains* and we compute the range of L^p-boundedness of the Bergman projection associated to these domains. More precisely, for a matrix $B \in M_n(\mathbb{Q})$ whose entry in the j-th row and k-th column is denoted b^j_k, the domains we study are of the form

$$\mathcal{U} = \left\{ z \in \mathbb{C}^n : \prod_{k=1}^{n} |z_k|^{b^j_k} < 1 \text{ for all } 1 \leq j \leq n \right\}.$$

We realize these domains as quotients of polydiscs by a finite group of biholomorphic automorphisms. We find that there is a positive integer $\kappa(\mathcal{U})$, which we call the complexity of $\mathcal{U}$, such that the Bergman projection is a bounded operator on $L^p(\mathcal{U})$ if and only if

$$\frac{2\kappa(\mathcal{U})}{\kappa(\mathcal{U})+1} < p < \frac{2\kappa(\mathcal{U})}{\kappa(\mathcal{U})-1}.$$

This so-called complexity is surprising in that it is determined by the arithmetic properties of the representative matrix, rather than the shape of the domain it represents.

Received: August 12, 2020

Coalescing Ballistic Annihilation

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Abstract of Report Talk: Ballistic annihilation is a stochastic model for chemical reactions first introduced in the 1980's statistical physics literature. In the model, particles are placed randomly throughout the real line and then proceed to move at fixed, preassigned velocities. Collisions result in mutual annihilation of both particles involved. Despite being analyzed to the satisfaction of physicists during the 1990's, the process has only recently bent to rigorous mathematical analysis. We generalize a breakthrough result from Haslegrave, Sidoravicius, and Tournier from 2019 to the setting in which collisions result in coalescence rather than mutual annihilation. For the three-velocity system and most symmetric coalescence rules, we pinpoint the smallest initial density of middle-speed particles at which infinitely many such particles survive. The proof uses recursion via a powerful, yet simple transformation known as the mass transport principle.

Received: August 12, 2020

Robust and Efficient Phase Retrieval from Magnitude-Only Windowed Fourier Measurements

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Abstract of Report Talk: We propose and analyze a new generalization of an existing algorithm to reconstruct a complex vector (up to a global phase factor) from the squared magnitude of its windowed discrete Fourier transform. This is more commonly referred to as a phase retrieval problem, since this process requires the recovery of critically important phase information from magnitude-only measurements. This is a challenging yet fascinating inverse problem since there are often several possible solutions. The proposed algorithm utilizes results from discrete Fourier analysis to linearize the governing equations and obtain a highly structured Fourier based linear system. This linear system of equations can be efficiently inverted using the fast Fourier transform (FFT) algorithm. This provides relative phase information which we use to construct a special class of banded matrices, on which we perform spectral analysis to retrieve individual phase information. In addition to developing an efficient reconstruction algorithm, we provide mathematically rigorous theoretical error bounds in the case of noisy measurements, and provide numerical simulations demonstrating that this algorithm is computationally efficient and able to recover data in the presence of noise.

Received: August 12, 2020

Juggling Coefficients in Complete Recurrent Sequences

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Williams College [Mentor:Steven Miller]

Abstract of Report Talk: A sequence of positive integers is complete if every positive integer is a sum of distinct terms. For example, powers of 2 are complete via binary decompositions. Conversely, powers of 10, i.e., decimal expansions, are not. Characterizing complete sequences has previously led to proving weaker forms of Goldbach's conjecture and to improvements on radix sorting. We focus our attention on the combinatorial aspect of complete sequences, which are numeration systems built on recurrent complete sequences.

We extend the work of Fraenkel, Gewurtz, and Merola by full characterization of a specific group of numeration systems defined by the Positive Linear Recurrence Sequences (PLRS). PLRS are integer sequences defined by a recurrence relation of the form $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L}$, with imposed initial conditions, which we denote as $[c_1, c_2, \ldots, c_L]$. For example, Fibonacci Sequence $F_{n+1} = F_n + F_{n-1}$, complete by the Zeckendorf's Theorem, is denoted $[1, 1]$. For sequences of this type, we prove that if a sequence generated by L coefficients is complete, then so is the sequence where the last coefficient is decreased. Hence, finding an upper bound on the last coefficient determines all complete sequences with the first $L - 1$ terms unchanged.

With this important tool at hand, we prove precise bounds for the largest N that preserves completeness for PLRS with generating coefficients $[1, \ldots, 1, 0, \ldots, 0, N]$ or $[1, 0, \ldots, 0, 1, \ldots, 1, N]$. These two families, even though they comprise a very small subset of all possible sequences, suggest the behavior of complete sequences when modifications to the coefficients are made. In our work we primarily employ Brown's Criterion, which states that a sequence $\{H_n\}_{n=1}^{\infty}$ is complete if and only if each term in the sequence is bounded above by the sum of all previous terms plus 1, i.e., $H_{n+1} \leq 1 + \sum_{i=1}^{n} H_i$. We then define Brown's Gap as the amount by which this inequality is not equal and use it to compare the growth of sequences with different coefficients. Finally, we conjecture precise bounds for the largest last coefficient that preserves completeness for other patterns of initial coefficients.

Received: August 12, 2020

A Tauberian approach to an analog of Weyl's Law for the Kohn Laplacian

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Abstract of Report Talk: In Riemannian geometry, Weyl's Law relates the growth of eigenvalues of the Laplace-Beltrami operator to geometric information about the underlying manifold. In particular, the leading coefficient in the asymptotic expansion is proportional to the volume of the manifold. For CR geometry, although no statement quite as simple as Weyl's Law is known, in 1984 Stanton and Tartakoff obtained an analog of Weyl's Law for eigenvalues of the Kohn Laplacian acting on $(0, q)$-forms ($q \geq 1$) on hypersurfaces in $\mathbb{C}^n$. A recent paper (Bansil and Zeytuncu 2019) found the leading coefficient in the asymptotic expansion for functions on spheres. In this talk we present a new computation of the leading coefficient using Karamata's Tauberian theorem. We conjecture that this representation can be generalized to an analog of Weyl's Law for functions on general CR manifolds.

Received: August 12, 2020

Removing Symmetry in Circulant Graphs and Point-Block Incidence Graphs

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Abstract of Report Talk: A vertex v in a graph G is *fixed* if it is mapped to itself under every automorphism of G. The fixing number of a graph G is the minimum number of vertices, when fixed, fixes all of the vertices in G. Fixing numbers were introduced by Laison and Gibbons, and independently by Erwin and Harary. Fixing numbers have also been called determining numbers by Boutin.
A *circulant graph* is a graph of n vertices in which the i-th vertex is adjacent to the $(i + j)$th and $(i - j)$th graph vertices for each j in a list L. We determine the fixing number for multiple classes of circulant graphs, showing in many cases the fixing number is 2. However, we show that circulant graphs with *twins*, which are pairs of vertices with the same open neighborhoods, have higher fixing numbers.
A *point-block incidence graph* is a bipartite graph $G = (P, B)$ with a set of point vertices $P = \{p_1, p_2, ..., p_r\}$ and a set of blocks $B = \{B_1, B_2, ..., B_s\}$ where $p_i \in P$ is adjacent to $B_j \in B \Leftrightarrow p_i \in B_j$. We show that symmetries in certain block designs cause the fixing number to be as high as $\frac{|V(G)|}{4}$. We also present several infinite families of graphs in which fixing any one vertex in G fixes every vertex in G, thus removing all symmetries from the graph.
This is joint work with Alvaro Carbonero and Joe Vargas.

Received: August 12, 2020

Optimal Addressing Schemes for Subfamilies of Block Graphs and Related Graphs

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Abstract of Report Talk: Graham and Pollak introduced the concept of a t-addressing of a graph as it applied to message routing in communication networks. A length t addressing of a graph G is a labelling of the vertices of G by t-tuples consisting of the symbols $a, b, 0$ such that the number of positions at which corresponding entries of their labels are distinct and nonzero equals the distance between the two labeled vertices. For a simple graph G, an optimal addressing $N(G)$ is the minimum t for which there exists a t-addressing of G. It has been shown that $h(D) \leq N(G) \leq |V(G)| - 1$, where $h(D)$ is the maximum of the number of positive and negative eigenvalues for the distance matrix D of G. An addressing is said to be eigensharp if $N(G) = h(D)$. In this talk, we identify several families of eigensharp graphs including various subfamilies of block graphs, prism graphs, and torus graphs. We also show the optimal addressing scheme for additional graph families such as flower graphs, spider graphs, wheel graphs, and stacked prism graphs.

Received: August 12, 2020

The Integer-magic Spectra of Trees

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Abstract of Report Talk: For any positive integer h, a graph $G = (V, E)$ is said to be h-magic if there exists a labeling $l : E(G) \to \mathbb{Z}_h - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \to \mathbb{Z}_h$ defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. The integer-magic spectrum of a graph G, denoted by $IM(G)$, is the set of all $h \in \mathbb{N}$ for which G is h-magic. So far, only the integer-magic spectra of trees of diameter at most five have been determined. In this paper, we determine the necessary and sufficient conditions for a tree, T, to be h-magic, and we provide a technique to determine the integer-magic spectra of a tree of any diameter.

Received: August 12, 2020

Classifying toric 3-fold codes of dimensions 4 and 5

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Abstract of Report Talk: A toric code, introduced by Hansen to extend the Reed-Solomon code as a k-dimensional subspace of $(\mathbb{F}_q^*)^m$, is determined by a toric variety or its associated integral convex polytope P, where $k = |P \cap \mathbb{Z}^m|$. Previous authors have classified toric surface codes with dimension up to $k = 7$. We classify all 4-dimensional toric codes of polytopes in $\mathbb{R}^3$ using White's description of polytopes with 4 lattice points, and we present progress toward the same for dimension 5 codes using work by Blanco and Santos. In particular, for $k = 4$ we first prove formulae for the minimum distances of codes coming from empty tetrahedra of the form

$$T(s,t) = conv\{(0,0,0),(1,0,0),(0,0,1),(s,t,1)\},$$

where $\gcd(s,t) = 1$, which occurs as a subpolytope for each subsequent case. We then uncover which disagreements in parameters s and t two polytopes may have while still yielding codes which are generated by matrices identical up to scaling and reordering their columns. Likewise, for $k = 5$ we prove bounds on the minimum distances of each class and indicate where uncertainty in a total classification remains.

Received: August 12, 2020

Quantitative error analysis of discontinuous Galerkin methods for the linear convection equation

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The Ohio State University [Mentor:Yulong XING]

Abstract of Report Talk: The discontinuous Galerkin (DG) method is a class of finite element numerical methods using discontinuous piecewise polynomial solution space and is widely applied for numerical simulations of partial differential equations. In this project, we use mathematical tools to provide a quantitative error estimate of the semi-discrete DG method for the linear convection equation on a periodic domain. Different choices of numerical fluxes are considered. One novel finding is that, using Fourier analysis, we are able to decompose the error term into physical and nonphysical components, and the effect of different fluxes on each component can be evaluated. We show that the nonphysical terms have a stronger influence on the short-time error, while the physical term will dominate the error in long time simulation. In addition, we verify a few existing convergence and superconvergence results with the Fourier approach for the upwind biased DG scheme, the DG scheme with central flux, and the energy-conserving scheme in Fu and Shu J. Comput. Phys. 394 (2019) 329-363. Extensions to specially constructed nonuniform meshes are also discussed. Numerical results are provided to validate the theoretical results.

Received: August 12, 2020

Split limiting behavior of random matrices with prescribed discrete spectra

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Abstract of Report Talk: Random matrix theory successfully models many systems, from the energy levels of heavy nuclei to zeros of L-functions. While most ensembles studied have continuous spectral distribution, Burkhardt et al introduced the ensemble of k-checkerboard matrices, a variation of Wigner matrices with entries in generalized checkerboard patterns fixed and constant. In this family, $N-k$ of the eigenvalues are of size $\sqrt{N}$ and were called bulk while the rest are tightly contrained around certain multiple of N and were called blip. We extend their work by allowing the fixed entries to take different constant values. We can construct ensembles with blip eigenvalues at any multiples of N we want with any multiplicity (thus we can have the blips occur at sequences such as the primes or the Fibonaccis). The presence of multiple blips creates technical challenges to separate them and to look at only one blip at a time. We overcome this by choosing a suitable weight function which allows us to localize at each blip, and then exploiting cancellation to deal with the resulting combinatorics to determine the average moments of the ensemble; we then apply standard methods from probability to prove that almost surely the limiting distributions of the matrices converge to the average behavior as the matrix size tends to infinity. For blips with just one eigenvalue in the limit we have convergence to a Dirac delta spike, while if there are k eigenvalues in a blip we again obtain hollow $k \times k$ GOE behavior.

Received: August 12, 2020

On Two-sided Matchings in Infinite Markets

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Abstract of Report Talk: Matching is a branch of economic theory that has seen real-life applications in the assignment of doctors to medical residencies, students to schools, and cadets to branches of military services. Although standard matching models are finite, economic theorists often lean on infinite market models as approximations of large market behaviors. While matching in finite markets has been studied extensively, the study of infinite matching models is relatively new. In this paper, we lift a number of classic results for one-to-one matching markets, such as group strategy proofness, comparative statics, and respect for unambiguous improvements, to infinite markets via the compactness theorem of propositional logic. In addition, we show that two versions of the lattice structure of finite markets carry over to infinite markets. At the same time, we prove that other results, such as weak Pareto optimality and strong stability property, do not hold in infinite markets. These results give us a clearer sense about which matching results are the most canonical.

Received: August 12, 2020

Lattice Models for Condensation in Levin-Wen Systems

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Abstract of Report Talk: Levin-Wen string-net models have been used to represent quantum double models, which are topological phase systems modeling quantum error correction methods. Topological phases of matter were introduced by Kitaev and others as a possible fault-tolerant platform for quantum computation, putting topological quantum fields at the intersection of physics and category theory. The most well-understood transition between topological phases of matter is anyon condensation. We write down lattice models which introduce an ancillary space to the string-net lattice and perform anyon condensation by introducing terms to the Levin-Wen Hamiltonian which can be tuned by a parameter. As particular examples, we demonstrate that this model is equivalent to condensation in known models for toric code and doubled semion.

Received: August 12, 2020

Partition rank vs geometric rank of tensors

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Abstract of Report Talk: A tensor is a higher order analogue of a matrix: a matrix gives a bilinear form (by specifying its coefficients), and a tensor gives a multilinear form. Tensors are much more complicated than matrices, because up to a change of basis there are finitely many $N \times N$ matrices but uncountably many $N \times N \times N$ tensors. We compare several notions of tensor rank to each other. The slice rank of an $N \times N \times N$ tensor takes a value between 0 and N, and has had stunning applications to combinatorics. The geometric rank of a tensor is a more recent notion, which generalizes the definition of matrix rank as the codimension of the kernel; it is generally smaller than the slice rank. We show that slice rank is upper bounded by twice the geometric rank, which indicates that these two notions are essentially equivalent. This is the first bound for the slice rank in terms of geometric rank. As an application, we prove that the slice rank of an $N \times N \times N$ tensor is linearly bounded by the analytic rank—prior to our work, the best bound was polynomial.

Received: August 12, 2020

Optimal Strategies for the Explorer-Director Game

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Abstract of Report Talk: The Explorer-Director game on a graph, G, includes two players with different roles. Starting at vertex v, the Explorer indicates a distance to move a token and the Director moves the token to a vertex the given distance away from the current position. The Explorer's goal is to maximize the number of distinct vertices visited while the Director's goal is to minimize this. In this game when both players play optimally, $f_d(G, v)$ is the number of distinct vertices visited throughout the game. An optimally played game is finished when $f_d(G, v)$ vertices have been visited. For cycle graphs, $f_d(G, v)$ was found by Nedev and Muthukrishnan in 2008 when they introduced the game.
We develop non-adaptive optimal strategies for the Explorer on some path graphs at various starting vertices. Furthermore, we describe a polynomial-time algorithm for determining when a game is finished under optimal play on any graph. Finally, we introduce the concept of a director-closed set and discuss its relation to completed games.

Received: August 12, 2020

New Helly-type theorems for diameter

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Abstract of Report Talk: In 1982, Bárány, Katchalski, and Pach proved that if the intersection of any $2d$ members of a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ has diameter 1, then $\bigcap \mathcal{F}$ has diameter at least d^{-2d}, and they conjectured that d^{-2d} could be replaced by $cd^{-1/2}$ for some constant $c > 0$. A consequence of Naszódi's 2016 breakthrough on the volumetric version of their conjecture is that if the intersection of any $d(d+3)/2$ sets of $\mathcal{F}$ have diameter at least 1, then $\operatorname{diam}(\bigcap \mathcal{F}) \geq cd^{-1}$. We prove, using only elementary concepts from convexity, a quadratic improvement on this statement: If the intersection of any $2d^2$ sets has diameter at least 1, then $\operatorname{diam}(\bigcup \mathcal{F}) \geq d^{-1/2}$. Using similar techniques, we prove a robust fractional version of Bárány, Katchalski, and Pach's conjecture, showing that if a large proportion of the $2d$-tuples of $\mathcal{F}$ intersect with diameter 1, then there is a large subset of $\mathcal{F}$ whose intersection has diameter at least $cd^{-1/2}$.

Received: August 12, 2020

Tropical Curves with Thin Newton Polygons

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Abstract of Report Talk: Tropical geometry is a piecewise-linear version of algebraic geometry. In the case of two dimensions, the key objects are tropical plane curves, a combinatorial analog of algebraic plane curves. These objects arise from algebraic curves through the process of tropicalization. A common phenomenon in tropical geometry is that the tropicalization of an object and the tropical analog of that object are related, but not identical.
We investigate instances of this phenomenon for tropical curves arising from thin Newton polygons, either having no interior lattice points or all interior lattice points collinear. Firstly, we ask: where can the intersection points of two algebraic curves map to in the intersection of their tropicalizations? Using resultants and the theory of non-Archimedean fields, we completely classify this in the case that the polygons are unimodular triangles. Our second investigation considers polygons with collinear interior points, which give rise to hyperelliptic algebraic curves, which in turn tropicalize to hyperelliptic metric graphs; however, not every possible set of edge lengths is achieved. Using the moduli theory developed by Brodsky, et al., we determine the possible edge lengths, thus measuring the obstruction to a hyperelliptic chain arising from the planar tropicalization of a hyperelliptic curve.

Received: August 12, 2020

On binomial coefficients associated with Sierpiński and Riesel numbers

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Moravian College [Mentor:Joshua Harrington]

Abstract of Report Talk: A Sierpiński (resp. Riesel) number is an odd integer k such that $k \cdot 2^n + 1$ (resp. $k \cdot 2^n - 1$) is composite for all $n \in \mathbb{N}$. Finch, Harrington and Jones considered Sierpiński numbers of the form $k = x^r + x + c$ for $r, c \in \mathbb{N}$. We investigate the existence of Sierpiński numbers and Riesel numbers as binomial coefficients through applications of covering systems. In particular, we show that the set of values of r for which there exist infinitely many Sierpiński numbers of the form $k = \binom{x}{r}$ has asymptotic density 1. A similar study is also performed on Riesel numbers. Additionally, generalizations to base a-Sierpiński numbers and base a-Riesel numbers are considered.

Received: August 12, 2020

Invariant and Prime Parking Sequences

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Abstract of Report Talk: Parking functions were first introduced by computer scientists Konheim and Weiss in 1966. A straightforward description of a parking function $\alpha \in [n]^n$ is that of n drivers in a linear parking lot, each with a preferred parking spot denoted by α_i, wishing to park on a one-way street with parking spots labeled $1, \ldots, n$. The drivers attempt to park using the following procedure:

1. Driver i parks at parking spot s_i if it is available;
2. if the driver's preferred parking spot is occupied, she parks at the first available spot after s_i; and
3. if there are none, she leaves the parking lot without parking.

Note that in the classical parking function, each car has length equal to 1; however, *parking sequences* are a generalization that allow the i-th car to have length $y_i \in \mathbb{Z}_+$ and a trailer T to park on the first $z-1$ spots of the street. A preference sequence in which all the cars are able to park without any collisions occurring is called a valid parking sequence.
Our work furthers the research of Adeniran and Yan (2020) on *invariant parking sequences* and we present a bijection between these objects and vector parking functions, which are a well-known extension of parking functions. We also define *prime parking sequences* which are a generalization of prime parking functions. Based on some preliminary data, we present a conjecture for the number of prime parking sequences given the fixed length vector $(1, ..., 1, k)$ for any $k \geq 2$.

Received: August 12, 2020

Covering Graphs and Linear Extensions of Signed Posets

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Abstract of Report Talk: In an effort to generalize Richard Stanley's theory of P-partitions, Victor Reiner defined posets in terms of root systems. This relates the directed graph interpretation of a poset to a set of vectors in $\mathbb{R}^n$. The beauty in this definition is that it generalizes easily to a notion of signed posets. In much of Stanley's work, he uses linear extensions of posets. However, there is no immediate generalization of a linear extension for signed posets. In order to accommodate this, Reiner uses the Jordan-Hölder set, which is an important object in the theory of finite lattices. By developing a notion of signed linear extensions of signed posets, our results prove the equivalence of the Jordan-Hölder set and a set of linear extensions of a covering graph as defined by Thomas Zaslavsky. This connects Reiner's work and the theory of signed graphs in an analogous fashion to the connection between Stanley's work and the theory of unsigned graphs.

Received: August 12, 2020

Efficient (j, k)-Domination on Chess Graphs

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Abstract of Report Talk: Graphs defined by the legal moves of a chess piece are a classical setting for efficient domination problems. For a graph G, a function $f : V(G) \to \{0, 1, \ldots, j\}$ is an *efficient* (j, k)*-dominating function* if, for all $v \in V(G)$,

$$\sum_{w \in N[v]} f(w) = k,$$

where $N[v]$ is the closed neighborhood of v (R.R. Rubalcaba and P.J. Slater. Efficient (j, k)-domination. *Discussiones Mathematicae Graph Theory*, 27(3): 409, 2007). While efficient $(1, 1)$-domination and efficient $(1, k)$-domination are well-studied, less is known about efficient (j, k)-domination.

We completely characterize the efficient (j, k)-dominating functions of King's graphs. Generalizing our results, we prove by construction that $G \boxtimes H$ is efficiently (j, k)-dominatable if and only if both G and H are. Additionally, we provide a necessary condition for efficient (j, k)-domination based on clique structure, following our observations on Bishop's graphs.

On the torus, the Queen's and Bishop's graphs are realizable as Cayley graphs. We apply character theory to determine the spectra of these graphs, through which we determine their efficient (j, k)-dominating functions.

For the standard $n \times n$ Queen's graph, we exploit an equitable partition to show computationally that for $4 \leq n \leq 226$, efficient (j, k)-domination occurs only when $n = 10$(!!). Expanding this approach, we construct an infinite class of graphs having an efficient (j, k)-dominating function from a common quotient graph over an equitable partition. This provides a partial answer to a question of Rubalcaba and Slater.

Received: August 12, 2020

2 Notions of Prime Graph Minimality

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Abstract of Report Talk: For a finite group G, its prime graph has the set of prime divisors of $|G|$ as its vertex set, and two vertices p, q are adjacent if there exists an element of order pq. Our focus is the minimal prime graphs $\Gamma(G)$ of finite solvable groups. By considering a slightly larger class of graphs which encodes a different notion of minimality, we establish novel construction methods and an enumerative connection to triangle-free, 3-colorable graphs which are maximal with respect to both properties. Considering the minimality property loops back to enable a strong classification of prime graphs of solvable groups by their diameter.

Received: August 12, 2020

The Maximum Likelihood Degree of n-cycle Models

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Abstract of Summary Talk: This project studies maximum likelihood estimation(MLE) from the viewpoint of algebraic geometry.
The maximum likelihood (ML) degree, or the number of critical points of the likelihood function, is a widely studied topic. In the literature, Hosten, Ketan, and Sturmfels introduced the scaling coefficients for each statistical model, and asked the following question: “How do scaling vectors change the ML degree?”
In this project, we show how to use scalings to reduce the ML degree of Binary-n-cycle models, a class of parametrized discrete exponentials encoded by nondecomposable undirected graphs. First, we introduce fixed-zero-submodels to reduce the upper bound of ML degree by applying the Bernstein-Khovanskii-Kushnirenko (BKK) theorem. Next, Geiger, Meek, Sturmfels stated that nondecomposable-undirected-graphs generally do NOT have a rational MLE. This leads to our main result: we use Kapranov's Horn Uniformisation to give a family of scaling vectors dropping the ML degree to 1 for arbitrary dimension, and we give the explicit form of the MLE as a rational polynomial.
To illustrate the results, we will provide constructive proofs that the ML degree of scaled binary-4-cycle-models is bounded by 64, where some fixed-zeroes-submodels can reduce the bound to 4, and some facial-unit scalings give ML degree exactly 1.

Received: August 12, 2020

Algebraic and Tropical Dimensions of Lattice Polygons

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Abstract of Report Talk: The moduli space $\mathcal{M}_g$ of algebraic curves of genus g is a principal object of study in algebraic geometry. In 2009, Castryck and Voigt introduced the locus $\mathcal{M}_P \subset \mathcal{M}_g$ of curves with Newton polygon P with g interior lattice points, and in 2015, Brodsky, et al., introduced its tropical analog $\mathbb{M}_P$. This polyhedral moduli space encodes all metric graphs of genus g of *tropical* curves arising from a Newton polygon P, and has the same dimension as its algebraic counterpart. Thus the value of $\dim(\mathbb{M}_P)$ is important from both a tropical and an algebro-geometric standpoint.
Our main result completely determines the possible values of $\dim(\mathbb{M}_P)$. We split into hyperelliptic polygons, which have all interior lattice points collinear, and nonhyperelliptic polygons, which have two-dimensional interiors. For a nonhyperelliptic polygon P, we prove that $g \leq \dim(\mathbb{M}_P) \leq 2g+1$ and that these bounds are sharp for $g \notin \{3,4,7\}$. Our main tool is a formula due to Coles, et al., which computes degrees of freedom in tropical curves from simple combinatorial properties of a dual triangulation. For a hyperelliptic polygon P, we develop a similar tool for computing the degrees of freedom and prove that $g \leq \dim(\mathbb{M}_P) \leq 2g-1$, with these bounds sharp for all g.

Received: August 12, 2020

Signed posets and a B-symmetric generalization of Stanley's acyclicity theorem

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Abstract of Report Talk: Richard P. Stanley's 1973 acyclicity theorem is a classic graph-theoretic result which states that the number of acyclic orientations of a graph G is $|\chi_G(-1)|$, where χ_G denotes the chromatic polynomial of G. We are interested in two particular generalizations of this fact. In 1982, Zaslavsky proved an acyclicity theorem for signed graphs using a signed version of the chromatic function. In 1995, Stanley used his chromatic symmetric function to prove an acyclicity theorem for unsigned graphs in which the number of sinks of the orientation is taken into account.
Our work aims to join these two generalizations. A *B-symmetric function* (in analogy with the finite Coxeter groups B_n) is a formal power series in variables $\{\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots\}$ which is invariant under *signed permutations* of the indices, those satisfying $\pi(-n) = -\pi(n)$. We use a B-symmetric generalization of Stanley's chromatic symmetric function for signed graphs developed by a past group of the same research program. In order to count acyclic orientations, we develop a new sort of quasi-B-symmetric function to obtain a result for signed posets analogous to Stanley's fundamental theorem of P-partitions.

Received: August 12, 2020

Products of Differentiation, Multiplication, and Composition on Discrete Weighted Banach Spaces

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Abstract of Report Talk: In recent years, discrete analogues to classical function spaces such as the Bloch space ($\mathcal{B}$), Hardy spaces (H^p), and weighted Banach spaces (H^∞_ν) have been developed. The study of operators on these spaces have included the multiplication, composition, and weighted composition operators. The discrete derivative has played a crucial role in the characterization of the fundamental concepts of boundedness and compactness of certain operators. In this work, we first study the discrete derivative as an operator acting on the discrete weighted Banach spaces. We characterize compactness, compute its operator norm, and show that it cannot be an isometry or a compact operator. Next, we study its interactions with multiplication and composition operators. We include some known results for completeness, then compute operator norms of products of these operators and characterize their properties, such as boundedness. The end goal is to understand the discrete derivative enough to develop discrete analogues of spaces such as the Hardy-Sobolev spaces (S^p).

Received: August 12, 2020

Ramsey Theory in Models of Set Theory where the Axiom of Choice Fails

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Abstract of Report Talk: Ramsey Theory aims to find large monochromatic sets under certain colorings. There are a lot of theorems in the Ramsey theory whose proof explicitly or implicitly uses the Axiom of Choice to pick non-constructive elements. This project focuses on Ramsey-theoretic statements in models of set theory where the Axiom of Choice fails. I am interested in two kinds of models: permutation models with atoms and symmetric submodels of forcing extensions, in both of which Choice fails because "symmetry" is required. The idea is to find symmetric colorings without symmetric monochromatic sets, or to show that every symmetric coloring has a symmetric monochromatic set in these models. Blass showed in 1977 that the infinite Ramsey theorem holds in the basic Fraenkel model and fails in the basic Cohen model, but Open Ramsey had not been investigated. In this project, I proved that Open Ramsey is true in the basic Fraenkel model and the infinite Ramsey theorem holds in the ordered Mostowski model. Also, the usual proof of Open Ramsey over the nonnegative integers given by Galvin and Prikry assumes the Axiom of Choice, and I gave an alternative proof without using it. The next step of my project is to study feeble filters, which are closely connected with the Ramsey property, as well as to study Open Ramsey in the basic Cohen model and the ordered Mostowski model. This project evaluates the level of choice in these models with respect to Ramsey-theoretic statements.

Received: August 12, 2020

Two Questions on Matching Complexes

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Abstract of Report Talk: A matching of a graph G is a set of edges that share no common end-points. The set of all matchings of G forms its matching complex $M(G)$, a simplicial complex in which each n-dimensional face of $M(G)$ represents a matching comprised of $n+1$ edges in G. We study two questions about matching complexes. First, we consider when a matching complex is a 2-dimensional Buchsbaum complex. The Buchsbaum condition is a topological property of simplicial complexes defined by certain local connectedness and global "purity" conditions. We describe families of graphs whose matching complexes are Buchsbaum by defining graph operations which preserve the Buchsbaum property. We also describe the local structure of all graphs whose matching complexes are 2-dimensional Buchsbaum complexes. Secondly, we study sequences of graphs generated by taking repeated matching complexes. For $k > 1$, the k^{th} matching of G is defined inductively as the matching complex of the edges in $M^{k-1}(G)$. We define $d(G)$ to be the smallest integer n such that $M^n(G) = \emptyset$; if there is no such integer then $d(G) = \infty$. The graphs C_5 and the net graph are the only graphs G for which the edges of $M(G)$ are isomorphic to G. If G is a graph such that $d(G) = \infty$, we show that there is some m such that $M^m(G)$ contains C_5 or the net graph as a subgraph. This allows us to completely describe all graphs with a finite d value. Additionally, we show that if a graph G has $d(G) = \infty$ and G is not C_5 or the net graph, then the number of edges in $M^k(G)$ grows without bound as k tends to infinity.

Received: August 12, 2020

On Bounds, Winning Strategies, and Generalizing the Zeckendorf Game

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Abstract of Report Talk: Zeckendorf proved that every positive integer n can be written uniquely as the sum of non-adjacent Fibonacci numbers. We use this to create a two-player game. Given a fixed integer n and an initial decomposition of $n = nF_1$, the two players alternate by using moves related to the recurrence relation $F_{n+1} = F_n + F_{n-1}$, and whoever moves last wins. The game always terminates in the Zeckendorf decomposition, though depending on the choice of moves the length of the game and the winner can vary. Previous work showed that for $n \geq 2$ Player 2 always has a winning strategy, though the proof is non-constructive, and the length of any game starting with n is at least of size n and of size at most $n \log n$. In our research, we tighten the bound on the game length to $\frac{\sqrt{5}+3}{2}n - \frac{\sqrt{5}+1}{2}Z(n) - IZ(n)$, where $Z(n)$ is the number of terms in Zeckendorf Decomposition $Z(n) = \Theta(\log n)$, $IZ(n)$ is the sum of indices in Zeckendorf Decomposition $IZ(n) = O(\log^2 n)$. Next, we expand the game to a p-player game with $p \geq 3$, and show that for any $p \geq 3$ player game, when $n \geq 5$, no player has the winning strategy. Some interesting results in multi-alliances situations and especially, 2-alliance situations are also discussed. Lastly, since a phenomenon similar to Zeckendorf Decomposition occurs to a non-constant coefficient recurrence $a_{n+1} = n\,a_n + a_{n-1}$. We extend the game based on this new recurrence and show that the game is finite and fair (playable).

Received: August 12, 2020

On Generalized Conjectures about Partitions with Congruence Relations and Difference Conditions

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Abstract of Report Talk: Integer partitions have long been of interest to number theorists, perhaps most notably Ramanujan, and are related to many areas of mathematics including combinatorics, modular forms, analysis, and mathematical physics. Here, we focus on partitions with gap conditions and partitions with parts coming from fixed residue classes. Let $\Delta_d^{(a,b)}(n) = q_d^{(a)}(n) - Q_d^{(b)}(n)$ where $q_d^{(a)}(n)$ counts the number of partitions of n into parts with difference at least d and size $\geq a$ and $Q_d^{(b)}$ counts the number of partitions into parts $\equiv \pm b \pmod{d+3}$. In 1956, Alder conjectured that $\Delta_d^{(1,1)}(n) \geq 0$ for all positive n and d. This conjecture was proved partially by Andrews in 1971, by Yee in 2008, and was fully resolved by Alfes, Jameson and Lemke Oliver in 2011. Alder's conjecture generalizes several well-known partition identities, including Euler's theorem that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts, as well as the first of the famous Rogers-Ramanujan identities.
In 2020, Kang and Park considered an extension of Alder's conjecture which relates to the second Rogers-Ramanujan identity by considering $\Delta_d^{(a,2,-)}(n) = q_d^{(a)}(n) - Q_d^{(2,-)}(n)$ where $Q_d^{(2,-)}(n)$ counts the number of partitions into parts $\equiv \pm 2 \pmod{d+3}$ excluding the $d+1$ part. Kang and Park conjectured that $\Delta_d^{(2,2,-)}(n) \geq 0$ for all $d \geq 1$ and $n \geq 0$, and proved this for d of the form $2^r - 2$ and n even.
We prove Kang and Park's conjecture for even d and even n by generalizing methods of Andrews and Yee. We further adapt work of Alfes, Jameson and Lemke Oliver to generate asymptotics for the related functions. Finally, we present a more generalized conjecture for higher $a = b$ for all positive n and d and prove it for infinite classes of n and d.

Received: August 12, 2020

Tinkering with Lattices: A New Take on the Erdos Distance Problem

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Abstract of Report Talk: The Erdős distance problem concerns the least number of distinct distances that can be determined by N points in the plane. The integer lattice with N points is known as *near-optimal*, as it spans around $O(N/\sqrt{\log(N)})$ distinct distances which is the lower bound for a set of N points (Erdős, 1946). The only previous non-asymptotic work relating to the Erdős distance problem that has been done was carried out for $N \leq 13$. We take a new non-asymptotic approach to this problem, studying the distance distribution, or in other words, the plot of frequencies of each distance of the $N \times N$ integer lattice. More specifically, we study the distance distributions of possible subsets of the integer lattice; although this is a restricted case, we find that the structure of the integer lattice allows for the existence of subsets which can be chosen so that their distance distributions have certain properties, such as emulating the distribution of randomly distributed sets of points for certain small subsets, or those of the larger lattice itself for certain geometric configurations.

With the motivation of characterizing the behavior of the distance distributions of subsets of the lattice, as compared to that of the full lattice, we define an error which compares the distance distribution of a subset with that of the full lattice. In particular, we calculate the error by scaling the frequency of each distance in the lattice subset to account for the size of the set, and then average the absolute difference between the scaled frequencies of the distance distribution of the subset and that of the full integer lattice. As the error shows how closely the distribution of a subset emulates that of the full lattice, we show that the structure of the integer lattice allows us to take subsets with certain geometric properties in order to both maximize and minimize error, by exploiting the potential for sub-structure in the integer lattice. We show these geometric constructions explicitly; further, for large N, we calculate explicit upper bounds for error for when $p = 4, 5$, and $\lceil \frac{N^2}{2} \rceil$ and prove the following lower bound for the error:

$$\text{Error} \geq \begin{cases} \frac{N^3}{N+2} + \frac{N^2}{N+2} - \frac{10N}{3(N+2)} & \text{if } p \leq \frac{\log_5(N)}{5}\left(11 - 2\sqrt{10}\right), \\ \frac{N^4}{8p^2} & \text{if } p > \frac{\log_5(N)}{5}\left(11 - 2\sqrt{10}\right). \end{cases}$$

Received: August 12, 2020

Constructions of Generalized MSTD Sets in Higher Dimensions

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Abstract of Report Talk: Let A be a set of finite integers and define

$$A + A \ = \ \{x + y : x, y \in A\}, \quad A - A \ = \ \{x - y : x, y \in A\}$$

and for non-negative integers s and d set

$$sA - dA \ = \ \underbrace{A + \cdots + A}_{s} - \underbrace{A - \cdots - A}_{d}.$$

A More Sums than Differences (MSTD) set is an A where $|A+A| > |A-A|$. Although MSTD sets had been found, it was initially thought that as n approaches infinity, the percentage of subsets of $[0, n]$ that are MSTD would go to zero as addition is commutative and subtraction is not. However, in a surprising 2006 result, Martin and O'Bryant proved that a positive percentage of sets are MSTD, although this percentage is extremely small, about 10^{-3} percent. This result was extended by previous REU students who showed that a positive percentage of sets are generalized MSTD sets, sets for $\{s_1, d_1\} \neq \{s_2, d_2\}$ with $|s_1A - d_1A| > |s_2A - d_2A|$, and that in d-dimensions, a positive percentage of sets are MSTD.

For many such results, establishing explicit MSTD sets in 1 dimensions relies on the specific choice of the elements on the left and right fringes of the set to force certain differences to be missed while desired sums are attained. In higher dimensions, the geometry forces a more careful assessment of what elements have the same behavior as 1-dimensional fringe elements. We study fringes in d-dimensions and use these to create new explicit constructions. We prove the existence of generalized MSTD sets in d-dimensions and the existence of k-generational sets, which are sets where $|cA + cA| > |cA - cA|$ for all $1 \leq c \leq k$. We then prove that under certain conditions, there are no sets such $|cA + cA| > |cA - cA|$ for all $c \in \mathbb{N}$.

Received: August 12, 2020

Numerical Range of Composition Operators on the Hardy Space of the Unit Ball

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Abstract of Report Talk: The numerical range of a bounded operator T on a Hilbert space is defined as

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

We explore the numerical range of a composition operator C_A on the Hardy space, $H^2(\mathbb{B}_n)$, of the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$, induced by an $n \times n$ matrix A that is a self-map of $\mathbb{B}_n$. The composition operator C_A is defined by

$$C_A f = f \circ A \quad \text{for all} \quad f \in H^2(\mathbb{B}_n).$$

We show that spaces of homogeneous polynomials of degree k, denoted $H_k(\mathbb{B}_n)$, are invariant under C_A. We find a matrix representation of C_A restricted to $H_k(\mathbb{B}_n)$. We then characterize the numerical range of C_A induced by a variety of matrices, including forward shift, backward shift, circular shift, anti-diagonal, and arbitrary permutation matrices. For example, we show that $W(A) \subseteq W(C_A)$ in general, and $W(A) = W(C_A)$ when A is the circular shift matrix. To achieve this, we compute the invariant subspaces of $C_A|_{H_k(\mathbb{B}_n)}$ and decompose the matrix representation of $C_A|_{H_k(\mathbb{B}_n)}$ as a direct sum of matrices. We then use combinatorial and computational methods and well-known numerical range results to characterize the numerical ranges of the composition operator.

Received: August 12, 2020

Zeros of Complex Random Polynomials Spanned by Bergman Polynomials

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Abstract of : We study the expected number of zeros of

$$P_n(z) = \sum_{k=0}^{n} \eta_k p_k(z),$$

where $\{\eta_k\}$ are complex-valued i.i.d standard Gaussian random variables, and $\{p_k(z)\}$ are polynomials orthogonal on the unit disk. When $p_k(z) = \sqrt{(k+1)/\pi} z^k$, $k \in \{0, 1, \dots, n\}$, we give an explicit formula for the expected number of zeros of $P_n(z)$ in a disk of radius $r \in (0, 1)$ centered at the origin. From our formula we establish the limiting value of the expected number of zeros, the expected number of zeros in a radially expanding disk, and show that the expected number of zeros in the unit disk is $2n/3$. Generalizing our basis functions $\{p_k(z)\}$ to be regular in the sense of Ullman–Stahl–Totik, and that the measure of orthogonality associated to polynomials is absolutely continuous with respect to planar Lebesgue measure, we give the limiting value of the expected number of zeros of $P_n(z)$ in a disk of radius $r \in (0, 1)$ centered at the origin, and show that asymptotically the expected number of zeros in the unit disk is $2n/3$.

Received: August 12, 2020

Generalizing the Explorer-Director Game

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Abstract of Report Talk: Consider a two-player game on a graph G with a token starting at a vertex v. Each turn, player one (Explorer) specifies a distance, then player two (Director) moves the token to a vertex the given distance away. Explorer aims to maximize the number of distinct vertices visited, while Director seeks to minimize this number. Let $f(G, v)$ be the number of vertices visited throughout the game when both players play optimally. This game was introduced in 2008 by Nedev and Muthukrishnan, who found $f(G, v)$ for cycle graphs. We prove novel upper and lower bounds for $f(G, v)$ for any graph along with exact formulas for trees, tree-like graphs, and square lattices in terms of their diameter. We also introduce two new variations of the game where the Director moves using paths or trails instead of geodesics. We found that knowing $f(G, v)$ for geodesics gives no information about the corresponding quantity under either variation.

Received: August 12, 2020

Spectral Properties of the Exponential Distance Matrix

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Abstract of Report Talk: Spectral graph theory is the study of the spectrum of a matrix representation of a graph. One common way to represent a graph G is with the distance matrix, which is defined entry-wise by letting the (u, v)-entry be $d(u, v)$, where $d(u, v)$ is the distance between the vertices u and v. However, if G is not connected, $d(u, v)$ may be infinite. An alternative is the exponential distance matrix, defined entry-wise by letting the (u, v)-entry be $q^{d(u,v)}$ where $|q| < 1$. Disconnected graphs can be represented by this matrix, as $q^{d(u,v)}$ will be 0 for u, v in different components. We establish several properties for the spectrum of this matrix, produce families of graphs which are uniquely determined by their spectrum, and construct some infinite cospectral families. This project was done as part of the 2019 Iowa State University REU program. Additional details can be found at arXiv:1910.06373.

Received: August 12, 2020

Local Properties of Difference Sets

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Abstract of Report Talk: Erdős and Shelah asked what we can learn about a large and complicated object X from properties that are satisfied by each small piece of X. We study the following variant of this problem, first studied by Erdős and Sós. Given a set of real numbers A, we consider the *difference set* $A - A = \{|a - b| : a, b \in A\}$. While a random set A is expected to have $|A - A| = \Theta(|A|^2)$, arithmetic progressions satisfy $|A - A| = \Theta(|A|)$.
Let $g(n, k, \ell)$ denote the minimum size of $|A - A|$, taken over all sets A of n numbers that satisfy the following local property: every subset $A' \subset A$ of k numbers satisfies $|A' - A'| \geq \ell$. Intuitively, every k numbers from A span many differences. We derive several new bounds for $g(n, k, \ell)$. We now state two of our results.
Erdős and others were interested in *linear thresholds* of local properties problems: the smallest ℓ for which the size of the global property is superlinear. We establish the linear threshold of the differences problem.

Theorem 1. *For every k, we have $g(n, k, k-1) = n - 1$ and $g(n, k, k) \gg n$.*

The following is the simplest of a family of bounds that we derive.

Theorem 2. *When k is a power of two, we have*

$$g\left(n, k, \frac{k^{\log_2(3)} + 1}{2}\right) = \Omega\left(n^{1+\frac{1}{k-1}}\right).$$

Received: August 12, 2020

Origami Hexagon Deformations and Flip Graph

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University of Michigan [Mentor:Harry Richman]

Abstract of Report Talk: By utilizing origami folding processes, researchers have been able to solve a wide range of engineering problems such as fabricating different robot morphologies. Leveraging origami structures in this way requires an understanding of the geometric and numerical properties of folding configurations. We studied the collection of configurations of an isolated hexagon consisting of six equilateral triangles around a vertex on the triangular lattice. We quantitatively characterized this moduli space parameterized by six fold angles. By applying second order approximation, we obtained the moduli space of deformations near the flat state as the variety Yin R^6,

$$Y = \begin{cases} x_5^2 = x_2^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3 \\ x_1 + x_2 = x_4 + x_5 \\ x_2 + x_3 = x_5 + x_6. \end{cases}$$

There are six natural flip operations on the space Y, corresponding to a local change at each of the six folds on the hexagon. We created an acyclic directed graph from the action of these flip operations on integer-valued points of Y. This flip graph is analogous to the Calkin–Wilf tree. The vertices of Calkin–Wilf tree correspond one-to-one to positive rational numbers, with root 1 and two children of $\frac{a}{b}$ defined as $\frac{a}{a+b}$, $\frac{a+b}{b}$.
By considering rotations and reflections, the flip graph has connected components which are isomorphic to each other, where each component is graded. We gave out closed formulas for the number of vertices on each level and the number of mountain-valley configurations among them. By generating the Markov transition matrix, we calculated the limiting distribution for mountain-valley configurations. On Pythagorean 4-tuples, there is an action by sign changes and the matrix A_4

$$A_4 = \begin{bmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 2 \end{bmatrix},$$

which generates a graph that has a precise relation to the flip graph.

Received: August 12, 2020

Eigenvalue Conditioning for Polynomial Eigenvalue Problems

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UC Santa Barbara [Mentor:Maribel Bueno]

Abstract of Report Talk: It is well-known that slightly perturbing the coefficients of a polynomial can greatly affect its roots. It is also known that appropriate representations of the polynomial in non-power bases, such as Chebyshev or Bernstein, may result in locally well-conditioned zeros. A natural generalization of the problem of computing polynomial roots is the polynomial eigenvalue problem (PEP). Given a matrix polynomial

$$P(x) = \sum_{i=0}^{d} P_i x^i, \quad P_i \in \mathbb{C}^{n\times n},$$

we look for numbers λ_0, called *eigenvalues*, such that $P(\lambda_0)y = 0$ with $y \neq 0$. Solving the PEP is essential to applications from finite element discretizations of continuous models, or as approximations to more general nonlinear eigenproblems, e.g. vibration analysis of buildings and machines. In our work, we consider matrix polynomials P written in the power basis, and compare the sensitivity of its eigenvalues to small changes in the matrix coefficients, as P is represented in non-power bases. Unexpectedly, we show that the conditioning for the power basis is almost always better than for Chebyshev, Lagrange, and Newton bases. We also give sufficient conditions for the non-power bases to perform comparably or better than the power basis, in terms of the relative position of the interpolation nodes and the eigenvalues. Our results are illustrated with numerical experiments.

Received: August 12, 2020

Solving Partial Differential Equations Using Radial Basis Function Interpolation

Steve R Manns (s-manns.1@onu.edu)
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Abstract of Report Talk: Following the "classical techniques", a class of numerical schemes for solving partial differential equations known as radial basis function collocation methods were developed. The attractiveness of these techniques lies in their simplicity; for example, an approach known as the Method of Approximate Particular Solutions (MAPS) reduces the challenge of solving a boundary value problem to solving for unknown coefficients by interpolation. Utilizing particular solutions that were derived previously, we demonstrate how accurate numerical solutions of certain boundary value problems can be obtained using the MAPS in MATLAB. Although considerable strides have been made in regard to deriving particular solutions for different radial basis functions, the uniqueness of this work is the emphasis placed on implementing the MAPS using computer programs.

Received: August 12, 2020

F-Signature and the Torsion Subgroup of the Divisor Class Group in Prime Characteristic Rings

Isaac K Martin (isaac.act.martin@gmail.com)
University of Utah [Mentor:Karl Schwede]

Abstract of Report Talk: The action of Frobenius is an important tool used to characterize the singularities of varieties associated to rings of primes characteristic. An invariant known as F-signature is used to measure how far a strongly F-regular singularity is from being a smooth point, and it is known to be positive if and only if the associated ring is strongly F-regular. In this talk, we discuss the history of F-singularities, Frobenius splitting, and F-signature. We also summarize our recent result; the F-signature of a local strongly F-regular ring R bounds the cardinality of the torsion subgroup of the divisor class group of R, and thereby provide a relationship between two seemingly unrelated invariants of local strongly F-regular rings.

Received: August 12, 2020

Properties of the Involution Derangement Graph

Lucy A Martinez (marti310@go.stockton.edu)
East Tennessee State University [Mentor:Anant Godbole]

Abstract of Report Talk: The derangement graph on all permutations is defined as the graph whose vertex set is the set of all permutations on $[n]$ with two vertices being adjacent if they have no position in common. Much is known about this graph, but specifically as a Cayley Graph whose vertex set is a group. Permutations have plenty of interesting subsets, which makes us question, what happens if we restrict the vertex set to one of these subsets? We examine the involution derangement graph, which is a subgraph of the permutation derangement graph. The involution derangement graph is the derangement graph whose vertex set is the set of all involutions on $[n]$, with an involution being a permutation whose inverse is itself. We study classic graph theory properties such as degree, diameter, chromatic number, independence number, and more and compare the properties of the involution derangement graph to those of the permutation derangement graph.

Received: August 12, 2020

Helly Numbers for Isometries of a Polytope

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Abstract of Report Talk: Helly's theorem is a foundational result in convex geometry. It states that if F is a finite family of convex subsets of $\mathbb{R}^d$ such that any subfamily $F' \subseteq F$ of size $d+1$ has nonempty intersection, then F has nonempty intersection. For a family W of subsets of $\mathbb{R}^d$, a quantitative Helly theorem for W is a statement of the following type: There is some $n \in \mathbb{N}$ such that for any finite family F of convex subsets of $\mathbb{R}^d$, if the intersection of any subfamily of size n contains an element of W, then $\bigcap F$ contains an element of W. If no quantitative Helly theorem holds for W, we say that W has infinite Helly number. In this work, we show that the set W of isometric copies of a fixed polytope in $\mathbb{R}^d$ has infinite Helly number, and the same holds for the set of affine copies of volume one of a fixed polytope in $\mathbb{R}^d$. Further, we use this to show optimality of past quantitative Helly-type results.

Received: August 12, 2020

Spectral theory of the Laplace-Beltrami operator on almost Abelian Lie groups

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University of California Santa Barbara [Mentor:Zhirayr Avetisyan]

Abstract of Report Talk: We study the spectral properties of the Laplace-Beltrami operator on a simply connected almost Abelian Lie group ,i.e. , a Lie group with a non-Abelian Lie algebra that has a codimension 1 Abelian Lie subalgebra. Previous work has found global group coordinates, identified the left invariant Riemannian metrics on these Lie groups, and reduced the spectral problem of the Laplace-Beltrami operator to that of a self-adjoint second order ordinary differential operator on a weighted $L^2(\mathbb{R}, \mu)$ space through Fourier-Laplace transform. Through the use of a group automorphism and an unitary transform, we begin by reducing this operator to an equivalent one dimensional Schrödinger operator on $L^2(\mathbb{R})$, whose potential has exponential behavior at infinity. We identify the spectrum of this operator and the orbits of the eigenfunctions under the action of group automorphisms.

Received: August 12, 2020

How Many Discrete Monotone Functions are There?

Hanna Mularczyk (hmularczyk@college.harvard.edu)
CUNY [Mentor:Guy Moshkovitz]

Abstract of Report Talk: We enumerate multivariate monotone functions $f : [n]^k \to \{0, \ldots, n\}$ where we define order coordinate-wise, so $(x_1, \ldots, x_k) \geq (y_1, \ldots, y_k)$ iff $x_i \geq y_i$ $\forall i$. Notably, the problem is equivalent to counting antichains in the product poset $[n]^{k+1}$, where again we define order coordinate-wise. The previous best result gives asymptotics when $k > n^8$. The Boolean $n = 2$ case is classically known as "Dedekind's problem" and can be solved by counting independent sets in the Hasse diagram of $[2]^k$, though it depends on certain graph regularity conditions that only hold for $n = 2$ and have, historically, been difficult to work around. Our new work adapts an approach by Kahn to general n (and, beyond this, a larger class of somewhat-regular graphs) by allowing for regularity conditions to only approximately be met. The proof requires both figuratively and literally thinking outside the box by adding extra vertices and edges to the diagram of the box $[n]^k$ to make it more regular. Then we take a recent bound by Zhao et al. on independent sets in irregular bipartite graphs and apply it inductively on our modified multi-partite graph. This allows us to broaden the previous $k > n^8$ result to k linear in n, though we suspect we can improve this to a logarithmic relation.

Received: August 12, 2020

Bounding the Zeroing Algorithm: A Tool for Investigating Linear Recurrence Relations

Jack G Murphy (jgm4@williams.edu)
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Abstract of Report Talk: Speakers: Jack Murphy

The "Zeroing Algorithm" – discovered by Martinez, Mizgerd, Miller and Sun – is a powerful tool for examining homogeneous linear recurrence relations. For a brief background, Zeckendorf's Theorem – which states that any positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers – has been generalized to any Positive Linear Recurrence Sequence (PLRS), which are the class of sequences arising from a linear recurrence of the form $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}$, with $c_i \geq 0$ integers, and $c_1, c_L > 0$. However, if we attempt to further generalize a PLRS into a ZLRS – essentially allowing the first s coefficients $c_1, \cdots, c_s = 0$ – difficulties arise in extending Zeckendorf's Theorem, for although existence of decomposition still holds, uniqueness of decomposition is lost. (Also note that generalized Zeckendorf decompositions are not restricted to using a term at most once, as is required for a complete sequence like the Fibonaccis.) Thus, it would be useful to be able to convert any ZLRS into a corresponding PLRS – for then, we would have a set of initial conditions for the ZLRS that would yield unique decompositions. Configured correctly, the Zeroing Algorithm can do just that. (For brevity, we forego an exposition of the algorithm, as it is more expedient to get a sense of its usefulness in generalizing Zeckendorf's Theorem to ZLRS's.)

There are challenges in utilizing the Zeroing Algorithm to convert a ZLRS. This is because the length L of the resultant PLRS recurrence, along with the size of its coefficients, is directly tied to the run-time. Hence, minimizing the run-time is essential to containing the size of the derived recurrence and thus also the number of initial conditions needed to render uniqueness of decomposition in the original ZLRS.

There is no known bound for the run-time of the Zeroing Algorithm, and even what might affect the run-time is open. All that is currently known is that the eventual behavior of the Zeroing Algorithm (whether or not it terminates) is determined by its initial configuration. However, we have found an explicit expression to relate the initial configuration to the run-time, which in effect allows us to estimate the run-time. This has been shown for ZLRS's of length 3 and 4, and we are progressing towards proving this relation in general for an arbitrary ZLRS. We also have obtained definitive experimental evidence exhibiting inverse/exponential behavior between the run-time and initial configuration, leading to conjectures for the true bound of the Zeroing Algorithm.

Received: August 12, 2020

ELLIPTIC DEDEKIND SUMS FOR DISCRETE SUBGROUPS $PSL(2, \mathbb{C})$

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Abstract of Report Talk: Dedekind sums are classical objects in number theory associated to $PSL(2, \mathbb{Z})$. They arise from the transformation law of the Dedekind η-function. In his paper, L. J. Goldstein defined a generalization of the Dedekind sums to discrete subgroups of $PSL(2, \mathbb{R})$. On the other hand, elliptic Dedekind sums are analogues of Dedekind sums for $PSL(2, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field. In this talk, we present a generalization of these Dedekind sums to discrete subgroups Γ of $PSL(2, \mathbb{C})$. From the Fourier expansion of the Eisenstein series $E_A(P, s)$ at a cusp ζ for Γ by Elstrodt, Grunewald and Mennicke, we obtain an analogue of Kronecker's first limit formula following the work of Goldstein. The limit formula will lead to an analogue of the Dedekind η-function. From its transformation formula, we define a new analogue of the Dedekind sums attached to Γ and ζ. This should have applications to computations of modular forms over imaginary quadratic fields.

Received: August 12, 2020

THE DEGENERATION OF THE HILBERT METRIC ON IDEAL PANTS AND ITS APPLICATION TO ENTROPY

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Abstract of Report Talk: Entropy is a single value that captures the complexity of a group action on a metric space. We are interested in the entropies of a family of ideal pants groups Γ_T, represented by projective reflection matrices depending on a real parameter $T > 0$. These groups act on convex sets Ω_{Γ_T} which form a metric space with the Hilbert metric. It is known that entropy of Γ_T takes values in the interval $(\frac{1}{2}, 1]$, however, it has not been proven whether $\frac{1}{2}$ is the sharp lower bound. Using Python programming, we generate approximations of tilings of the convex set in the projective plane and estimate the entropies of these groups with respect to the Hilbert metric. We prove a theorem that, along with the images and data produced by our code, suggests that the lower bound is indeed sharp. This theorem regards the degeneration of the Hilbert metric on the convex set Ω_{Γ_T}.

Received: August 12, 2020

Clique Partitions of Random Graphs

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Abstract of Report Talk: A clique partition of a graph G is a collection of edge-disjoint cliques of G which contain each edge of G exactly once. A natural question to ask is: what is the "smallest" clique partition of G? We shall look at some variations of this question for the n-vertex binomial random graph $G(n, p)$, where each of the $\binom{n}{2}$ possible edges is inserted independently with probability p.
In this talk we show that the minimum size of a clique partition of $G(n, p)$ is with high probability $\Theta(n^2/(\log n)^2)$ when the edge-probability p is a constant. The proof is based on a randomized algorithm which decomposes the edge-set of $G(n, p)$ into edge-disjoint cliques. If time permits, we shall also discuss extensions of this result that (a) allow for non-constant p and (b) minimize other notions of size associated with clique partitions.

Received: August 12, 2020

Enumeration of Minimal 2-Cuts on Surfaces

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Carleton College [Mentor:Rob Thompson]

Abstract of Report Talk: Embed a graph on a surface. If cutting along the graph splits the surface into exactly two pieces, and every cut is necessary to split the surface, we call the graph embedding a minimal 2-cut. Our research describes a method to find all graphs that admit minimal 2-cuts on a genus g torus. Using topological methods, we restrict the collection of possible graphs that admit minimal 2-cuts. We begin by using the Euler characteristic to restrict the number of edges and vertices in such a graph. Graphs are further restricted by considering the properties of embeddings of subgraphs. For each graph that meets the topological restrictions, we use a modified version of a rotation system (a combinatorial encoding of a specific embedding) to confirm the existence of a minimal 2-cut. We demonstrate this process by constructing all minimal 2-cuts on the genus 1 and genus 2 torus.

Received: August 12, 2020

The Regular Isotopy Classes of Links Attainable from Thompson's Groups

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Abstract of Report Talk: In 2014, Vaughan Jones developed a method to produce links from elements of Thompson's group F and showed that all links arise this way. He also introduced a subgroup $\vec{F}$ of F and a method to produce oriented links from elements of this subgroup. In 2018, Valeriano Aiello showed that all oriented links arise from this construction. We classify exactly those regular isotopy classes of links that arise from F, as well as exactly those regular isotopy classes of oriented links that arise from $\vec{F}$, answering a question asked by Jones in 2018.

Received: August 12, 2020

The finite blocking problem and periodic points on the regular n-gon and double n-gon

Rafael M Saavedra (rmsaavedra@uchicago.edu)
University of Michigan [Mentor:Paul Apisa]

Abstract of Report Talk: Consider a ball bouncing in a polygonal billiards table. Given two points P and Q, the finite blocking problem asks whether there is a finite set of points S in the interior of the billiards table such that all billiards trajectories from P to Q pass through S.

Billiards trajectories on a rational polygon unfold to straight lines on a closed surface with a singular flat metric, a so-called translation surface. In 2017, Apisa and Wright showed that the finite blocking problem can be solved in part by determining which points on a translation surfaces are special points called periodic points.

In our work, we determine the periodic points on the regular n-gon and double n-gon translation surfaces. We use facts about the dynamics of horocycle flow and arguments in plane geometry to restrict the periodic points on these surfaces. This solves the finite blocking problem on the $(\pi/2, \pi/n, (n-2)\pi/(2n))$ and $(2\pi/n, (n-2)\pi/(2n), (n-2)\pi/(2n))$ triangles.

Received: August 12, 2020

Prime Walks to Infinity in $\mathbb{Z}[\sqrt{2}]$

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Abstract of Report Talk: We cannot walk to infinity along the prime numbers with bounded step size because we can find arbitrarily large gaps. To see this, let $p\#$ be the product of all primes at most p, then $p\# + 2$, $p\# + 3$, $\ldots$, $p\# + p$ are all composite, so we have a gap of length at least $p - 2$ for any arbitrarily large prime p. However, the question of whether it is possible to walk to infinity along the Gaussian primes is open and known as the Gaussian Moat problem. Motivated by the clustering of primes along asymptotes through the origin, we modify this problem to look at primes in the real quadratic integer ring $\mathbb{Z}[\sqrt{2}]$. We build a probability model of these primes according to their norm $a^2 - 2b^2$ by adapting the Prime Number Theorem and appealing to a combinatorial theorem for counting the number of lattice points in the region $|a^2 - 2b^2| \leq n^2$. We hope to use this model to make probabilistic statements about the existence of a prime walk. We also prove that the number of norms needed is infinite and perform a few moat calculations for various step sizes.

Received: August 12, 2020

On the t-Target Pebbling Conjecture

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Abstract of Report Talk: Graph pebbling is a network optimization model for satisfying vertex demands with vertex supplies (called pebbles), with partial loss of pebbles in transit. The pebbling number of a demand in a graph is the smallest number for which every placement of that many supply pebbles satisfies the demand.
The t-Target Conjecture (Herscovici-Hester-Hurlbert, 2009) posits that the largest pebbling number of a demand of fixed size t occurs when the demand is entirely stacked on one vertex. This truth of this conjecture could be useful for attacking many open problems in graph pebbling, including the famous conjecture of Graham (1989) involving graph products. It has been proven for complete graphs, cycles, cubes, and trees. In this paper we consider 2-paths, split graphs, and Kneser graphs, important classes of graphs in graph structure theory, graph coloring, and algorithms.
Using recently developed cost-related methods and induction, we prove the t-Target Conjecture for all 2-paths, split graphs of minimum degree 3, and Kneser graphs with $k = 2$ and $m \geq 5$, and build tools potentially useful for attacking other graphs as well, such as, we believe, more general classes of chordal graphs.

Received: August 12, 2020

Prime Graphs of Several Classes of Groups

Dawei Shen (shen.dawei@wustl.edu)
Texas State University [Mentor:Thomas Keller]

Abstract of Report Talk: The prime graph of a finite group G, also known as the Gruenberg-Kegel graph, is the graph with vertex set {primes dividing $|G|$} and edges p-q if and only if there exists an element of order pq in G. In 2015, *Gruber*, *Keller*, *Lewis*, *Naughton*, and *Strasser* constructed an orientation of complements of prime graphs of solvable groups and used it to show that they are exactly the triangle free and 3-colorable graphs.
By further studying properties of this orientation, we characterize prime graphs of the following classes of groups.
(1) Any class $\mathcal{C}$ such that {Groups of square-free order} $\subseteq \mathcal{C} \subseteq$ {Metanilpotent groups}
(2) Given any $n \in \mathbb{N}$, solvable groups of n^{th}-power-free order
(3) Groups of cube-free order
We also prove a necessary condition on complement prime graphs of groups whose composition factors are cylic or A_5. The condition is surprisingly neat and almost as strong as "triangle free and 3-colorable." As a corollary, these groups satisfy *N. V. Maslova*'s conjecture from *Unsolved Problems in Group Theory.* Namely, their prime graphs must be 3-colorable if it is triangle free. Our method involves orienting the complement prime graph of these possibly non-solvable groups.

Received: August 12, 2020

Structural Szemeredi–Trotter theorem for Lattices

Olivine Silier (osilier@caltech.edu)
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Abstract of Report Talk: The study of point-line incidences was initiated by Erdős. A point and a line form an *incidence* if the point is on the line. For a point set P and a line set L we denote the number incidences as $I(P, L)$. When $|P| = |L| = n$, the Szemerédi–Trotter theorem states that $I(P, L) = O(n^{4/3})$. The theorem is tight since there exist configurations with $\Theta(n^{4/3})$ incidences. The Szemerédi–Trotter theorem is very useful, having applications in combinatorics, number theory, computer science, harmonic analysis and more.

While the Szemerédi–Trotter theorem has been known for nearly four decades, hardly anything is known about the *structural problem*: characterizing the configurations with $\Theta(n^{4/3})$ incidences. Using an energy variant recently introduced by Rudnev and Shkredov, we derive a variety of structural properties for the case where the point set is a Cartesian product. Note that this is the case for all known maximal configurations. Our main result:

Theorem 3. *Let $\alpha \in (0, \frac{1}{2}]$, $|A| = n^{1-\alpha}$, $|B| = n^{\alpha}$. If $I(A \times B, L) = \Theta(n^{4/3})$ then there exists $k = \Omega(n^{\alpha})$ such that L contains $\Omega(\frac{n^{1+\alpha}}{k^2 \log(n)})$ families of at least k parallel lines.*

The two known configurations with $\Theta(n^{4/3})$ incidences show that the theorem is tight up to the $\log(n)$ factor.

Received: August 12, 2020

Optimality of Helices on Some 6-D Lie Groups

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Abstract of Report Talk: We consider a family of geometric optimal control problems on the six-dimensional Lie groups $SE(3)$, $SO(4)$ and $SO(1,3)$. We show that some extremals correspond to helices in the 3-D spaces $\mathbb{R}^3$, $\mathbb{S}^3$, and $\mathbb{H}^3$, and use necessary conditions for optimality to present an explicit parameterization of all of these helical extremals. Additionally, we prove that these helices are locally optimal for a finite length and we use Jacobi's sufficient condition to compute this critical length. Finally, we derive a scaling property relating the length at which helices loose optimality and the twisting rate along the helix, and we use this to compute and visualize the boundary between optimal and non-optimal helices.

Received: August 12, 2020

Projective Geometric Algebra for Paraxial Geometric Optics

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Abstract of Summary Talk: Geometric Algebra (GA), a subset of Clifford Algebra, is useful in many fields of study including optics. For decades GA was overshadowed by Gibbs' formulation of vector algebra, but researchers are finally understanding its power. GA offers conceptual clarity by granting equal representation to points, lines, and planes. As a result, geometric operations like rotations and reflections are represented by simple multiplication. GA also provides the complete framework for different types of algebras including imaginary numbers, quaternions, and differential forms. While these newly revived math tools are gaining acceptance in computer graphics and mechanical engineering, they have sparingly been applied to optics, or for our research, paraxial geometric optics, which employs specific approximations. In our research, we use the dual representation of projective geometric algebra (PGA) to show how basic optical systems in paraxial geometric optics can be described by simple expressions more efficiently than traditional ray tracing methods. We also use the outermorphism property of GA to show how ray transfer matrices can used to map object points onto images and can transform any geometric primitive just as easily. From our research, we conclude that PGA is a more natural mathematical description for paraxial geometric optics than ray tracing.

Received: August 12, 2020

CR-embeddability of quotients of the Rossi sphere via spectral theory

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Abstract of Report Talk: CR manifolds are smooth manifolds which generalize the notion of boundaries of domains in $\mathbb{C}^n$. Many abstract CR manifolds cannot be globally embedded as CR submanifolds of $\mathbb{C}^n$ for any n (Burns and Epstein 1990) but there are few well-known explicit examples of non-embeddable CR manifolds. The Rossi sphere, which is defined as the regular sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ endowed with a perturbed CR structure, is the canonical example of a non-embeddable abstract CR manifold. A modern result states that one can detect CR-embeddability for certain CR manifolds by analyzing the bottom of the spectrum of the Kohn Laplacian. In this talk, we study similar questions for spherical 3-manifolds that are obtained by taking the quotient of the sphere by left actions of finite subgroups of $SU(2)$. Using spectral-theoretic techniques, we prove that the quotient of the Rossi sphere by the antipodal map is CR-embeddable. We further generalize this result, proving that a quotient of the Rossi sphere which can be understood as a lens space $L(p, p-1)$ with a perturbed CR structure is CR-embeddable if and only if p is even. This characterization gives an infinite family of explicit examples of non-embeddable CR manifolds.

Received: August 12, 2020

Degree Sequence Realizations with Neighbor Constraints

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Abstract of Report Talk: Given a finite sequence of non-negative integers $D = (d_1, \ldots, d_n)$, the *degree sequence realization problem* asks if there exists a labeled simple graph G such that $v_i \in V(G)$ has degree d_i. This question can be answered in polynomial time using the Havel-Hakimi algorithm or the Erdos-Gallai theorem. Given a pair of finite sequences of non-negative integers $D = (d_1, \ldots, d_n)$ and $F = (f_1, \ldots, f_n)$, the *degree and neighbor sum problem* asks if there is a labeled graph G such that $v_i \in V(G)$ has degree d_i and the sum of the degrees of v_i's neighbors is f_i. Of particular interest is when the sequences D and F have tree realizations. This problem has natural motivations in chemistry, as nuclear magnetic resonance spectroscopy (NMR) can be used to compute the degree and neighbor sum sequences for carbon backbones of alkanes, which have a tree structure. Results on realizations can provide tools for identifying alkanes from NMR spectra. We establish necessary and sufficient conditions for a given degree sequence D and neighbor sum sequence F to have a tree realization, and also discuss an algorithm to produce such realizations.

Received: August 12, 2020

Delta Sets of Geometric Nonminimally Generated Numerical Monoids

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Abstract of Report Talk: A numerical monoid is a subset of the whole numbers with addition as a binary operation, and we can factor elements of a numerical monoid into its generating elements, which form a generating set. When a nonminimal element is included in the generating set, the structure of factorizations and some factorization invariants, in particular the delta set, change. In minimal geometric generating sets of the form $a^n, a^{n-1}b, ..., ab^{n-1}, b^n$, the delta set is simply $b - a$. We study the case when a nonminimal element, labeled s, is added to the generating set. In this specialized instance, we prove several basic properties of the delta set, characterize a majority of the case where $n = 2$ and $3(b-a) \leq l_s - 1$, and completely characterize the case where $n = 2$ and $b - a | l_s - 1$.

Received: August 12, 2020

Extending Virus Dynamics to k-level Starlike Graphs

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Abstract of Report Talk: In the COVID-19 era, virus propagation models are of great interest. Becker, Greaves-Tunnell, Kontorovich, Miller, Ravikumar, and Shen determined the long term evolution of virus propagation behavior on a hub-and-spoke graph of one central node and n neighbors, with edges only from the neighbors to the hub (a 2-level starlike graph), under a variant of the discrete-time SIS (Susceptible Infected Susceptible) model. The behavior of this model is governed by the interactions between the infection and cure probabilities, along with the number n of 2-level nodes. They proved that for any n, there is a critical threshold relating these rates, below which the virus dies out, and above which the probabilistic dynamical system converges to a non-trivial steady state (the probability of infection for each category of node stabilizes). For a, the probability at any time step that an infected node is not cured, and b, the probability at any time step that an infected node infects its neighbors, the threshold for the virus to die out is $b \leq (1 - a)/\sqrt{n}$.

We have extended this model to 3-level starlike graphs (connecting n_2 spoke nodes to the n_1 spoke nodes connected to the central hub), whereby n_2 3-level nodes are accounted for in addition to n_1 2-level nodes. This yields a critical convergence threshold of similar form to the 2-level case, which is $b \leq (1 - a)/\sqrt{n_1 + n_2}$.

Our analysis generalizes to k-level starlike graphs for $k \geq 3$ (each $k - 1$-level node has exactly n_k neighbors, and the only edges added are from the k-level nodes) for infection rates below the critical threshold of $(1 - a)/\sqrt{n_1 + n_2 + \cdots + n_{k-1}}$, and we will report on ongoing investigations.

Received: August 12, 2020

Disjointness of Actions of $SL(2, \mathbb{Z}_2)$ and $SL(2, \mathbb{F}_2[x])$ on Serre Trees

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Abstract of Report Talk: Serre (1980) showed that, for a discrete valuation field K, the group $GL(2, K)$ acts on an infinite regular tree T_K with vertex degree determined by the residue degree of K. Since the fields $\mathbb{Q}_p$ and $\mathbb{F}_p((x))$ have the same residue degree, the groups $GL(2, \mathbb{Q}_p)$ and $GL(2, \mathbb{F}_p((x)))$ act on isomorphic trees, and this isomorphism allows us to directly compare the actions of these two groups. In particular, we may ask whether pairs of actions from these two groups are ever conjugate as tree automorphisms.
Fixing $p = 2$, we developed a computer program to compute group actions by $GL(2, \mathbb{Q}_2)$ and $GL(2, \mathbb{F}_2((x)))$, and analyzed permutations induced on finite vertex sets. We show a permutation classification result for actions by $SL(2, \mathbb{Z}_2)$ and $SL(2, \mathbb{F}_2[x])$, and prove that actions by $SL(2, \mathbb{Z}_2)$ and $SL(2, \mathbb{F}_2[x])$ are not conjugate, with a finite set of exceptions. Lastly, we discuss how our methods could be applied when $p > 2$.

Received: August 12, 2020

Analytic Approaches to Completeness of Generalized Fibonacci Sequences

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Williams College [Mentor:Steven Miller]

Abstract of Report Talk: A sequence of positive integers is complete if every positive integer can be written as a sum of its terms, using each term at most once. As each number has a binary expansion, the sequence $\{H_n\}$ defined by $H_1 = 1$ and $H_{n+1} = 2H_n$, whose terms are simply the powers of 2, is complete; due to Zeckendorf's Theorem (every integer can be written uniquely as a sum of non-adjacent Fibonacci numbers) we see the slower growing sequence of Fibonacci numbers ($F_1 = 1,\ F_2 = 2,\ F_{n+1} = F_n + F_{n-1}$) is complete as well. We generalize these results and investigate completeness questions for Positive Linear Recurrence Sequences (PLRS), which are defined by special initial conditions and a recurrence relation of the form $H_{n+1} = c_1H_n + c_2H_{n-1} + \ldots + c_LH_{n-L+1}$, with integer $c_i \geq 0$ and $c_1, c_L > 0$.
To each PLRS $\{H_n\}$ we associate its characteristic polynomial $p(x) := x^L - \sum_{i=1}^{L} c_i x^{L-i}$, which always has exactly one positive root r. We develop a method of characterizing completeness of a PLRS by analyzing r, which we call its principal root; a famous example is the Golden Ratio $\phi = \left(1+\sqrt{5}\right)/2$, the principal root of the Fibonacci Sequence. We show that any sequence with principal root $r > 2$ is necessarily incomplete; further, through a series of analytical and combinatorial arguments, we determine a precise second bound $B_L \approx (L/2)^{2/L}$, which satisfies $1 < B_L < 2$ and $\lim_{L\to\infty} B_L = 1$, so that any sequence defined by L coefficients $c_1, \ldots, c_L$ with principal root $r < B_L$ is necessarily complete. Since this leaves us with an interval $[B_L,\ 2]$ containing roots of both complete and incomplete sequences, we conclude by showing that the set of principal roots of complete sequences and the set of principal roots of incomplete sequences are both dense in $(1, 2]$; in particular, for any $\varepsilon > 0$ there exists an L such that any ε-ball in $[B_L, 2]$ contains the root of both a complete and an incomplete sequence with L recurrence coefficients $c_1, \ldots, c_L$. This yields a full characterization of the roots of complete and incomplete sequences, allows us to effectively generate complete and incomplete sequences, and gives us a quick method for checking completeness of an arbitrary sequence, as bounding the size of roots is computationally simpler than verifying completeness directly.

Received: August 12, 2020

Computation of Eigenfunctions on Various Domains

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Abstract of Report Talk: Finding the eigenfunctions and eigenvalues of the Laplacian and Laplace-Beltrami operator is fundamental to solving a variety of differential equations commonly used in physics.
The paper develops a numerical method for computing eigenvalues and eigenfunctions of the Laplace and Laplace-Beltrami operators defined on arbitrarily shaped domains.
I have extended an approach previously used to solve the time-independent Schrödinger equation for a quantum particle in a two-dimensional infinite potential well by modifying it to solve for eigenfunctions and eigenvalues of the Laplacian and the Laplace-Beltrami operator with Dirichlet boundary conditions on general surfaces.
The method is based on embedding the given domain into a rectangular domain and substituting the infinite-potential well with a very large potential well.
A rectangle was chosen because the eigenfunctions are already known, but for the Laplace-Beltrami operator, I used different basis functions. The numerically calculated eigenvalues were compared with known analytical results for certain domains to evaluate the accuracy of the method. The method was coded in Matlab.

Received: August 12, 2020

A new approach to solve broken stick problems

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Abstract of Report Talk: Break a stick at random at $n-1$ points to obtain n pieces. We give an explicit formula for the probability that every choice of k segments from this broken stick can form a k-gon, generalizing similar work. The method we use can be applied to other geometric probability problems involving broken sticks, which are part of a long-standing class of recreational probability problems with several applications to real world models. Its main feature is the use of order statistics on the spacings between order statistics for the uniform distribution applied to the broken stick problems. We also present a discrete approach that can shed light on currently out of reach problems.

Received: August 12, 2020

Bifurcations in the construction of elastic Mobius bands

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Abstract of Report Talk: We apply optimal control theory to study the stable equilibrium shapes of a Möbius band that is modeled as an anisotropic elastic rod. Although previous work has focused on determining the equilibrium shapes of a Möbius band, little is known about the mechanics of how a circular rod deforms into a Möbius band as its ends are twisted. In this talk, we will show that a circular rod can experience a pitchfork bifurcation during this deformation. Using the necessary and sufficient conditions for optimality from optimal control theory, we numerically analyze how the critical twist angle at which these bifurcations occur depends on the stiffness parameters of the rod. Then, for a specific choice of stiffness parameters, we explore the structure of these bifurcations. Our findings show that although the elastic rod can bifurcate before being twisted into a Möbius band, all solution branches resulting from the bifurcation converge to the same Möbius band shape.

Received: August 12, 2020

Bounding Graph Gonality and Scramble Number

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Abstract of Report Talk: Divisor theory on algebraic curves is a facet of classical algebraic geometry used to determine how an algebraic curve embeds in different dimensions. In 2007, Baker and Norine defined divisor theory on finite graphs using "chip-firing" games. Since then, results about divisors on graphs have been lifted to results about curves. From this, divisorial graph gonality is defined as the smallest possible degree of a positive rank divisor on a graph. In 2019, Gijswijt, Smit, and van der Wegen proved that computing graph gonality is NP-hard. Thus, finding ways to bound graph gonality is integral to this new area of study. In 2020, Harp, Jackson, Jensen, and Speeter introduced a new graph invariant, scramble number, and proved it is a lower bound on gonality, strictly better than another bound, treewidth. While treewidth is minor monotone, scramble number is not; however, we show that scramble number is immersion minor monotone, and therefore yields a forbidden immersion minor characterization. We also explicitly find all forbidden immersion minors for graphs with scramble number at most two. Additionally, we explicitly construct new bounds on gonality for various families of graphs. Our proofs are constructive, generating effective divisors or scrambles to bound gonality.

Received: August 12, 2020

Symplectic and Contact Forms On Almost Abelian Lie Groups

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Abstract of Report Talk: An almost Abelian group is a Lie group that possesses a codimension one Abelian subgroup. A symplectic structure on a manifold is a closed, nondegenerate, differential 2-form. It is clear from the definitions that such a manifold must be even dimensional. For odd dimensions, the closely related structure is called a contact structure. In this project, we study properties of invariant symplectic and contact structures on almost Abelian Lie groups. On a connected almost Abelian group, a basis of invariant vector fields can be computed explicitly in terms of global group coordinates and any invariant tensor can be expanded in a global frame basis with constant coefficients. Here, we impose the extra conditions of symplecticity or contactness and solve the resulting equations. The class of almost Abelian Lie groups is rather wide and representative, and gives context for developing methods of non-commutative analysis on solvable Lie groups — a subject that is still little understood. Additionally, almost Abelian Lie groups (and algebras) are related with areas such as integrable systems, linear dynamical systems, and even theoretical cosmology.

Received: August 12, 2020

Virtual Knots and Artin Groups

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Abstract of Report Talk: One of the most famous invariants studied in the area of knot theory is the so-called knot group. This notion can be extended to be an invariant of virtual knots, a class of knots which contain crossings with no under or over-crossing information. It is known that some groups are not the group of any classical knot, but are the group of some virtual knot. In a paper by Se-Goo Kim, an algorithm was proposed for generating virtual knots whose knot group is a given group. This algorithm relies upon the group admitting a special type of presentation called a realizable Wirtinger presentation. It is well known that the only braid group which is isomorphic to a knot group is the braid group on three strands. Our group worked to determine Artin groups, a generalization of braid groups, which admit such a presentation, and when they do, we worked to calculate the virtual knots corresponding to various such groups, such as the Artin group corresponding to the Coxeter group H_2, which has presentation

$$\langle t_1, t_2 \mid t_1t_2t_1t_2t_1 = t_2t_1t_2t_1t_2 \rangle$$

In general, we have shown that any Artin group with either two or three generators, and with relations of odd length, is the group of some virtual knot. We have also shown that any such group with relations of even length is not the group of any virtual knot. They may, however, be the group of some virtual link.

Received: August 12, 2020

Properties of Fibotomic Polynomials

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Moravian College [Mentor:Joshua Harrington]

Abstract of Report Talk: Fibonacci polynomials are defined recursively in the following manner: $F_0(x) = 0$ and $F_1(x) = 1$, and for all $n \geq 2$, $F_n(x) = F_{n-2}(x) + xF_{n-1}(x)$. In this talk, we consider the irreducible factors of Fibonacci polynomials, which are called the Fibotomic polynomials. The Fibotomic polynomials are known to share similar root structures with cyclotomic polynomials, which make them an especially interesting class of polynomials to study. We prove several analogous properties for the Fibotomic polynomials that are well-known for the cyclotomic polynomials. Our investigation includes the study of the discriminants of the Fibotomic polynomials, as well as the factorization of Fibotomic polynomials modulo an odd prime.

Received: August 12, 2020

Moment estimation for lattice point discrepancy using Fourier analysis

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Abstract of Report Talk: Define $f(\vec{x}, R)$ to be the number of lattice point in the circle of radius R shifted by $\vec{x} = (x_1, x_2) \in \mathbb{Z}^2$ from origin minus the area of the circle, i.e.:

$$f(\vec{x}, R) = \#\{(j,k) \in \mathbb{Z}^2 : (j - x_1)^2 + (k - x_2)^2 \leq R^2\} - \pi R^2.$$

It is conjectured that $f(\vec{x}, R) = \mathcal{O}(R^{\frac{1}{2}+\varepsilon})$ for any fixed $\epsilon > 0$, which is known as Gauss's circle problem. In this paper, we study the higher moments of this error term with respect to shifts, which analyze the error term's underlying probabilistic distribution. Define the k-th power discrepancy as

$$\int_{\mathbb{T}^2} f(\vec{x}, R)^k d\vec{x}.$$

Huxley's 2014 paper showed the fourth power discrepancy is at most $\mathcal{O}(R^2 \log R)$. In this paper, using techniques of Fourier analysis and convolution, we find a new, simpler proof for Huxley's argument and show that any generalized bounded convex domain, whose boundary is smooth and has nowhere vanishing Gaussian curvature, has fourth power discrepancy $\mathcal{O}(R^2 \log R)$. Using this technique, we also obtain the L^p norm discrepancy for annuli of area $R^{1/2}$ for $2 \leq p < 4$.

Received: August 12, 2020

Optimal rates of exponential ergodicity for Markov chains on strings

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Abstract of Report Talk: Let $\mathcal{A}$ ($\mathcal{A}$ for "alphabet") be a finite set of cardinality at least two, and let $\mathcal{S} = \bigcup_{M=0}^{\infty} \mathcal{A}^M$ be the $\mathcal{A}$-valued sequences of finite length. Motivated by preexisting models of DNA evolution, we study a certain family of continuous-time Markov chains with state space $\mathcal{S}$. Each of these Markov chains is ergodic, meaning there is a unique invariant distribution, call it Π, and the transition probabilities $P_t(x, \cdot) \to \Pi$ (converge to Π) as $t \to \infty$. Using tools from optimal transport, we derive an explicit formula for the convergence rate ε_1 of $P_t(x, \cdot) \to \Pi$. In addition, we build on existing results to show that $\varepsilon_1 = \lambda_1$, where λ_1 is the spectral gap, for any reversible ergodic continuous-time Markov chain. This additional result may be of general interest, and in our setup it has some information-theoretic implications, which we briefly discuss. Finally, after introducing three real-world models of DNA evolution and discussing implications of our results for these models, we outline directions of future research concerning alternative models and statistical inference.

Received: August 12, 2020

A New Frobenius Template in a Matrix Ring

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Abstract of Report Talk: Frobenius problems, also known as "Chicken Nugget" problems, seek to find the largest nonnegative integer such that every integer after it can be written as a linear combination of the coprime generators using nonnegative integer coefficients. We now generalize Frobenius problems from a topic in number theory to a topic in ring theory. Previous to this work, research on generalized Frobenius problems has concentrated on the Gaussian integers and the rings $\mathbb{Z}[\sqrt{m}]$, where m is a square-free positive integer. We launch the study of Frobenius problems in commutative rings of 2×2 upper triangular matrices with constant diagonal. Using properties of matrix rings and modular arithmetic, we determine for which lists of matrices with integer and real number coefficients the Frobenius set is non-empty. Additionally, for each list such that the Frobenius set is non-empty, we determine the range of the Frobenius set. For the lists of two matrices, we find the construction of every matrix in the Frobenius set. For the lists of more than two matrices, we find the conditions under which the construction can be extended. Finally, we introduce a new species of Frobenius problems for future studies.

Received: August 12, 2020